Space-times admitting neutrino fields with zero energy and momentum

1973 J. Phys. A: Math. Nucl. Gen. 6 915

(http://iopscience.iop.org/0301-0015/6/7/011)

J. Phys. A: Math., Nucl. Gen., Vol. 6, July 1973. Printed in Great Britain. © 1973

Space–times admitting neutrino fields with zero energy and momentum

C D Collinson and P B Morris
Department of Applied Mathematics, University of Hull, Hull HU6 7RX, UK

Received 10 January 1973

Abstract. All space–times admitting a neutrino field having a zero energy–momentum tensor are found. One of the space–times is shown to admit two distinct neutrino fields.

A neutrino field is described by a two component spinor ψ^A satisfying Weyl's equation

$$\partial_{A\dot{C}}\psi^A = 0. \tag{1}$$

The energy–momentum tensor of the neutrino field is

$$E_{ij} = \mathrm{i}\{\sigma_i{}^{\dot{A}B}(\bar{\psi}_{\dot{A};j}\psi_B - \bar{\psi}_{\dot{A}}\psi_{B;j}) + \sigma_j{}^{\dot{A}B}(\bar{\psi}_{\dot{A};i}\psi_B - \bar{\psi}_{\dot{A}}\psi_{B;i})\} \tag{2}$$

where $\sigma_i{}^{\dot{A}B}$ are the generalized Pauli matrices and the semicolon denotes covariant differentiation. The space–time admitting the neutrino field as a source is found by solving Einstein's field equation with the expression (2) on the right-hand side.

Griffiths (1972) has investigated the class of space–times for which E_{ij} vanishes. He shows that these space–times are of Petrov type N or D. In a previous paper (Collinson and Morris 1972) the authors solved the Einstein–neutrino field equations under the assumption that the energy–momentum tensor takes the form

$$E_{ij} = \Lambda^2 l_i l_j$$

where l_i is the neutrino flux vector. Clearly those space–times for which E_{ij} vanishes can be deduced by putting Λ equal to zero. The resulting metric for the space–times of Petrov type N is

$$\mathrm{d}s^2 = -F\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r - \tfrac{1}{2}\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{3}$$

where $\partial F/\partial r = \partial^2 F/\partial z \partial\bar{z} = 0$. This space–time is well known, it is the general plane fronted gravitational wave. The neutrino flux vector is $l^i = A(u)\partial x^i/\partial r$ where A is an arbitrary function of u.

The metric for the space–times of Petrov type D (and possibly 0) is

$$\mathrm{d}s^2 = -\left(\frac{2\mu_0}{r}\right)\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r - \left(\frac{r^2}{2}\right)\mathrm{d}z\,\mathrm{d}\bar{z}, \tag{4}$$

where μ_0 is a constant.† In fact a nonzero μ_0 can be reduced to unity by means of the

† Collison and Morris (1972) contains a misprint. The last two metric components in equation (3.23) should read $-2/r^2$.

transformation $r \to (\mu_0)^{1/3} r$, $u \to (\mu_0)^{-1/3} u$, $z \to (\mu_0)^{-1/3} z$. The metric is then

$$\mathrm{d}s^2 = -\left(\frac{2}{r}\right)\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r - \left(\frac{1}{2r^2}\right)\mathrm{d}z\,\mathrm{d}\bar{z}, \tag{5}$$

and the neutrino flux vector is

$$l^i = r^{-2} A(u) \frac{\partial x^i}{\partial r} \tag{6}$$

where A is again an arbitrary function of u. The metric (3) is invariant in form under the transformation

$$r = r', \qquad u = \tfrac{1}{2} r'^2 - u'. \tag{7}$$

Under this transformation

$$l^{i'} = r'^{-1} A(\tfrac{1}{2} r'^2 - u')\left(\frac{\partial x^{i'}}{\partial u'} + r'^{-1}\frac{\partial x^{i'}}{\partial r'}\right).$$

We deduce that the original space–time (5) admits two distinct neutrino fields, one with the neutrino flux vector (6) the other with the neutrino flux vector

$$l^i = r^{-1} B(\tfrac{1}{2} r^2 - u)\left(\frac{\partial x^i}{\partial u} + r^{-1}\frac{\partial x^i}{\partial r}\right) \tag{8}$$

where B is an arbitrary function of $\frac{1}{2}r^2 - u$. Notice that repetition of the transformation (7) yields the identity transformation and so only *two* distinct neutrino fields are obtained. The authors are at present completing a study of those space-times which admit two distinct neutrino fields.

If μ_0 vanishes then the metric (4) becomes

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}r - \left(\frac{r^2}{2}\right)\mathrm{d}z\,\mathrm{d}\bar{z} \tag{9}$$

which is *flat*. The neutrino flux vector is again given by the expression (6). Since the transformation (7) no longer leaves the metric invariant one cannot generate the second neutrino field with flux vector given by the expression (8). The calculations in Collinson and Morris (1972) are incomplete for flat space–times and so for such space–times other solutions of the Weyl equation representing neutrino fields with zero energy–momentum tensor might exist. For example Wainwright (1972, private communication) has shown that the wave-like example given by Griffiths (1972) is also flat.

Note added in proof. The metrics obtained in this paper have also been obtained by J Wainwright in an unpublished note.

References

Collinson C D and Morris P B 1972 *Int. J. theor. Phys.* **5** 293–301
Griffiths J B 1972 *Commun. math. Phys.* **28** 295–9